SOME PROBLEMS OF THE THEORY OF HEAT CONDUCTION FOR A TWO-LAYERED MEDIUM

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Solutions—in complex and real form—are obtained for the nonstationary one-dimensional and stationary two-dimensional problems of the theory of heat conduction in a two-layered medium. The inversion formula of a certain integral transform is employed.

This note is concerned with the temperature distribution in a medium consisting of an infinite plate ($0 < x < l$) and a half-space ($l < x < \infty$) with different thermal properties.

In examining the one-dimensional nonstationary problem it is assumed that the initial temperature is arbitrary and that on the surface $x = 0$ it is required to satisfy a boundary condition of the third kind of general form. The solution of this problem, obtained in the form of real quadratures, is associated with the expansion in eigenfunctions of a certain spectral problem. In §2 below this expansion is used to find the two-dimensional steady-state thermal regimes in a layered medium.

§1. To solve the above problem of nonsteady temperature distribution it is necessary to integrate the system of equations

$$\frac{\partial^2 T}{\partial x^2} = \beta_1 \frac{\partial T}{\partial t}, \quad 0 < x < l, \quad t > 0,$$

$$\frac{\partial^2 T}{\partial x^2} = \beta_2 \frac{\partial T}{\partial t}, \quad l < x < \infty, \ t > 0 \qquad (1)$$

with the initial condition

$$T(x, 0) = f(x), \ 0 < x < \infty, \qquad (2)$$

the boundary conditions

$$aT_x(0, t) - bT(0, t) = F(t), \ T(\infty, t) < \infty, \ t > 0, \qquad (3)$$

and the temperature and heat flux continuity conditions

$$T(l-0, t) = T(l+0, t),$$
$$T_x(l-0, t) = \nu T_x(l+0, t). \qquad (4)$$

Applying the Laplace transformation

$$\bar{T}(x) = \int_0^\infty T(x, t) \exp(-pt)\, dt,$$

we arrive at the following boundary value problem:

$$\bar{T}'' = \beta_k [p\bar{T} - f(x)], \ k = 1, 2,$$

$$a\bar{T}'(0) - b\bar{T}(0) = \bar{F}; \ \bar{T}(\infty) < \infty,$$

$$\bar{T}(l-0) = \bar{T}(l+0), \ \bar{T}'(l-0) = \nu \bar{T}'(l+0). \qquad (5)$$

After a number of calculations the solution of problem (5) can be represented in the following form:

$$\bar{T}(x) = \frac{1}{\sqrt{p}\,\omega(p)} \int_0^\infty \Phi(x, \xi, p) f(\xi)\, d\xi + \bar{S}(x), \qquad (6)$$

where

$$\Phi = \begin{cases} \sqrt{\beta_1}\varphi(\xi, p)\left[\operatorname{sh}\sqrt{\beta_1 p}(l-x) + \delta \operatorname{ch}\sqrt{\beta_1 p}(l-x)\right], \\ \qquad 0 < \xi < x < l, \\ \sqrt{\beta_2}\varphi(x, p)\exp\left[-\sqrt{\beta_2 p}(\xi - l)\right], \\ \qquad 0 < x < l < \xi < \infty, \\ \delta\sqrt{\beta_1}\varphi(\xi, p)\exp\left[-\sqrt{\beta_2 p}(x-l)\right], \\ \qquad 0 < \xi < l < x < \infty, \\ \sqrt{\beta_2}\psi(x, p)\exp\left[-\sqrt{\beta_2 p}(\xi - l)\right], \\ \qquad 0 < l < x < \xi < \infty, \end{cases}$$

$$\varphi(\xi, p) = b \operatorname{sh}\sqrt{\beta_1 p}\,\xi + a\sqrt{\beta_1 p}\operatorname{ch}\sqrt{\beta_1 p}\,\xi,$$

$$\psi(x, p) = (b \operatorname{sh}\sqrt{\beta_1 p}\, l +$$
$$+ a\sqrt{\beta_1 p}\operatorname{ch}\sqrt{\beta_1 p}\, l)\operatorname{ch}\sqrt{\beta_2 p}(x-l) +$$
$$+ \delta(b\operatorname{ch}\sqrt{\beta_1 p}\, l + a\sqrt{\beta_1 p}\operatorname{sh}\sqrt{\beta_1 p}\, l)\times$$
$$\times \operatorname{sh}\sqrt{\beta_2 p}(x-l), \qquad (7)$$

and for $x, \xi < l$ and $x, \xi > l$ $\Phi(x,\xi,p) = \Phi(\xi,x,p)$,

$$\bar{S} = \begin{cases} -\dfrac{\bar{F}}{\omega(p)}\left[\operatorname{sh}\sqrt{\beta_1 p}(l-x) + \delta\operatorname{ch}\sqrt{\beta_1 p}(l-x)\right], \\ \qquad 0 < x < l, \\ -\dfrac{\bar{F}}{\omega(p)}\,\delta\exp\left[-\sqrt{\beta_2 p}(x-l)\right], \quad l < x < \infty, \end{cases}$$

$$\omega(p) = b(\operatorname{sh}\sqrt{\beta_1 p}\, l + \delta\operatorname{ch}\sqrt{\beta_1 p}\, l) +$$
$$+ a\sqrt{\beta_1 p}(\operatorname{ch}\sqrt{\beta_1 p}\, l + \delta\operatorname{sh}\sqrt{\beta_1 p}\, l),$$

$$\delta = \frac{1}{\nu}\sqrt{\frac{\beta_1}{\beta_2}}, \ \operatorname{Re}\sqrt{p} > 0. \qquad (8)$$

The application to solution (6) of the Riemann-Mellin inversion formula gives the unknown T(x, t):

$$T(x,\ t) =$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\exp(pt)}{\sqrt{p}\,\omega(p)}\, dp \int_0^\infty \Phi(x,\ \xi,\ p) f(\xi)\, d\xi +$$

$$+ S(x,\ t). \tag{9}$$

As may be seen from relation (9), the first term is associated with the inhomogeneity of the initial and the second with the inhomogeneity of the boundary condition, it being convenient to represent the function S(x, t) in the following form:

$$S(x,\ t) = \frac{d}{dt} \int_0^t F(t - \tau)\, u(x,\ \tau)\, d\tau, \tag{10}$$

having expressed it in terms of the solution u(x, t) of problem (1)–(4) with $f(x) \equiv 0$ and $F(t) \equiv 1$. The function u(x, t) is represented, in its turn, by complex integrals of the type

$$u = \frac{-1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \times$$

$$\times \left[\operatorname{sh} \sqrt{\beta_1 p}\,(l - x) + \delta \operatorname{ch} \sqrt{\beta_1 p}\,(l - x)\right] \frac{\exp(pt)}{p\,\omega(p)}\, dp,$$

$$0 < x < l,$$

$$u = \frac{-1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \delta \exp\left[-\sqrt{\beta_2 p}\,(x - l)\right] \frac{\exp(pt)}{p\,\omega(p)}\, dp,$$

$$l < x < \infty. \tag{11}$$

In reducing integrals (11) to real form it is possible to proceed in two ways. One of these, leading to a solution convenient for small values of t, consists in expanding the transformed solution in powers of $((\delta - 1)/(\delta + 1))((a(\beta_1 p)^{1/2} - b)/(a(\beta_1 p)^{1/2} + b))$. Such a solution was obtained by A. V. Luikov in [1] for the case of a boundary condition of the first kind ($a = 0$) $f(x) \equiv 0$ and $F(t) \equiv$ const.

The other method, leading to a solution effective at large times, requires the reduction of complex integrals such as (9) and (11) to real form by integration along a cut made along the negative part of the real axis of the plane of the complex variable p. Using Cauchy's theorem,* we obtain the function u(x, t) in real form:

$$u(x,\ t) = -\frac{1}{b} + \frac{1}{\pi} \int_0^\infty \eta(x,\ \lambda) \frac{\delta \exp(-\lambda t)}{\lambda \Delta(\lambda)}\, d\lambda. \tag{12}$$

Here,

$$\eta(x,\ \lambda) = \begin{cases} -i\varphi(x,\ -\lambda), & 0 < x < l, \\ -i\psi(x,\ -\lambda), & l < x < \infty, \end{cases} \tag{13}$$

$$\Delta(\lambda) = \omega(-\lambda)\,\omega^*(-\lambda). \tag{14}$$

Applying an analogous method to the first term in (9), we finally obtain

$$T(x,\ t) =$$

$$= \frac{1}{\pi} \int_0^\infty f(\xi)\, d\xi \int_0^\infty H(x,\ \xi,\ \lambda) \frac{\exp(-\lambda t)}{\sqrt{\lambda}\,\Delta(\lambda)}\, d\lambda +$$

$$+ S(x,\ t), \tag{15}$$

where

$$H = \begin{cases} -\delta\sqrt{\beta_1}\,\varphi(x,\ -\lambda)\,\varphi(\xi,\ -\lambda), & x,\ \xi < l, \\ -\sqrt{\beta_2}\,\varphi(x,\ -\lambda)\,\psi(\xi,\ -\lambda), & \\ & 0 < x < l < \xi < \infty, \\ -\delta\sqrt{\beta_1}\,\varphi(\xi,\ -\lambda)\,\psi(x,\ -\lambda), & \\ & 0 < \xi < l < x < \infty, \\ -\sqrt{\beta_2}\,\psi(\xi,\ -\lambda)\,\psi(x,\ -\lambda), & x,\ \xi > l. \end{cases} \tag{16}$$

§2. Setting t = 0 in (15), we obtain

$$f(x) = \frac{1}{\pi} \int_0^\infty f(\xi)\, d\xi \int_0^\infty H(x,\ \xi,\ \lambda) \frac{d\lambda}{\sqrt{\lambda}\,\Delta(\lambda)}, \tag{17}$$

which after certain transformations assumes the following form:

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\eta(x,\ \mu^2)\, d\mu}{\Delta(\mu^2)} \int_0^\infty f(\xi)\, r(\xi)\, \eta(\xi,\ \mu^2)\, d\xi, \tag{18}$$

where

$$r(\xi) = \begin{cases} \delta\sqrt{\beta_1}, & 0 < \xi < l, \\ \sqrt{\beta_2}, & l < \xi < \infty. \end{cases} \tag{19}$$

It is easy to see that Eq. (18) is the expansion of a function $f(x)$ of some class, specified on the interval $(0, \infty)$, into an integral in eigenfunctions of the singular spectral problem

$$\eta'' + \mu^2 \beta_k \eta = 0,\ k = 1,\ 2,$$

$$a\eta'(0) - b\eta(0) = 0,\ \eta(l - 0) = \eta(l + 0),$$

$$\eta'(l - 0) = \nu\eta'(l + 0),\ \eta(\infty) < \infty, \tag{20}$$

with a continuous spectrum of eigenvalues.

We will show that the expansion obtained makes it possible to construct an exact solution of the following two-dimensional stationary problem of the theory of heat conduction:

$$\Delta T(x,\ y) = 0,\ 0 < x < \infty,\ 0 < y < H,$$

$$aT_x(0,\ y) - bT(0,\ y) = 0,\ T(\infty,\ y) < \infty,$$

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*It can be shown that the equation $\omega(p) = 0$ does not have roots satisfying the condition $\operatorname{Re}\sqrt{p} > 0$.

$$T(l-0;\ y) = T(l+0;\ y),$$

$$T_x(l-0;\ y) = \nu T_x(l+0;\ y),$$

$$T(x,\ 0) = 0,\ T(x,\ H) = f(x). \tag{21}$$

Solving problem (21) by separation of variables, we arrive at the formula

$$T(x,\ y) = \int_0^\infty A(\mu) \frac{\operatorname{sh} \mu y}{\operatorname{sh} \mu H} \eta(x,\ \mu^2)\, d\mu. \tag{22}$$

Setting y = H in (22), we obtain

$$f(x) = \int_0^\infty A(\mu)\, \eta(x,\ \mu^2)\, d\mu, \tag{23}$$

whence on the basis of (18) with $\beta_1 = \beta_2 = 1$ we immediately find an expression for the unknown $A(\mu)$:

$$A(\mu) = \frac{2}{\pi \Delta(\mu^2)} \int_0^\infty f(\xi)\, r(\xi)\, \eta(\xi,\ \mu^2)\, d\xi. \tag{24}$$

In conclusion we note that the proposed method of solving stationary problems can be transferred without substantial modification to the case in which boundary conditions of the second and third kind are specified at y = 0 and y = H.

NOTATION

$\beta_k = c_k\rho_k/\lambda_k$; c_k is the specific heat; ρ_k is the density; λ_k is the thermal conductivity (k = 1 for $0 < x < l$, k = 2 for $l < x < \infty$); $\nu = \lambda_2/\lambda_1$.

REFERENCE

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